EdWest

School
Mock Examination, 2011

Question/Answer Booklet

MATHEMATICS
3C/3D Specialist
Section One:
Calculator free

Name: Solutions

Time allowed for this section
Reading time before commencing work: 5 minutes
Working time for this section: 50 minutes

Material required/recommended for this section
To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid, ruler, highlighters

Special items: nil

Important note to candidates
No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available
Section One: Calculator-free	6	6	50	40
Section Two Calculator-assumed	10	10	100	80
				120

Question	1	2	3	4	5	6	Total
Marks	6	10	9	5	4	6	40
Awarded							

Instructions to candidates

1. The rules for the conduct of Western Australian external examinations are detailed in the *Year 12 Information Handbook 2011*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

3. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that you **do not use pencil** except in diagrams.

Section One: Calculator-free (40 Marks)

This section has **six (6)** questions. Answer **all** questions. Write your answers in the space provided.
Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

Suggested working time for this section is 50 minutes.

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1. **[6 marks]**

Given $z = -2+i$, $w = 3e^{\frac{\pi}{4}i}$ and $v = cis\left(-\frac{\pi}{6}\right)$, determine as exact values:

(a) $3z + 6\bar{z}$ $= 3(-2+i) + 6(-2-i)$ [1]

$= -6 + 3i - 12 - 6i$

$= -18 - 3i$ ✓

(b) $(vw)^4$ $= \left(cis\left(-\frac{\pi}{6}\right) 3\,cis\left(\frac{\pi}{4}\right)\right)^4$ [3]

$= \left(3\,cis\,\frac{\pi}{12}\right)^4$ ✓

$= 81\,cis\,\frac{\pi}{3}$

$= 81\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= \frac{81}{2} + i\,\frac{81\sqrt{3}}{2}$ ✓

(c) The modulus and argument of $-i^3v$ [2]

$-i^3 v = -i^3 cis\left(-\frac{\pi}{6}\right)$

$= i\,cis\left(-\frac{\pi}{6}\right)$

$= cis\left(\frac{\pi}{2}\right) cis\left(-\frac{\pi}{6}\right)$

$= cis\left(\frac{\pi}{3}\right)$

modulus $= 1$ ✓ argument $= \frac{\pi}{3}$ ✓

2. **[10 marks]**

(a) Find:

(i) $\frac{d}{dx}\int_1^{2x}\frac{1-u^2}{\sin u}\,du$ **[2]**

$$= \frac{1-(2x)^2}{\sin 2x}(2)$$

(ii) $\int \tan x \, dx = \int \frac{\sin x}{\cos x}$ **[2]**

$$= -\ln|\cos x| + c$$

(b) Use an appropriate substitution to evaluate the integral: **[6]**

$\int_0^3 \sqrt{9-x^2}\,dx$

let $x = 3\cos u$

$\frac{dx}{du} = -3\sin u$

when $x=0$ $u=\frac{\pi}{2}$

$x=3$ $u=0$

$$= \int_{\frac{\pi}{2}}^{0} \sqrt{9-(3\cos u)^2}\,(-3\sin u\,du)$$

$$= \int_{\pi/2}^{0} \sqrt{9-9\cos^2 u}\,(-3\sin u\,du)$$

$$= \int_{\pi/2}^{0} \sqrt{9\sin^2 u}\,(-3\sin u\,du$$

$$= \int_{\pi/2}^{0} -9\sin^2 u\,du$$

$$= \int_{\pi/2}^{0} -9(1-\cos^2 u)\,du$$

$$= \int_{\pi/2}^{0} -9+9\left(\frac{1+\cos 2u}{2}\right)du$$

$$= \int_{\pi/2}^{0} \frac{-9}{2}+\frac{9\cos 2u}{2}\,du$$

$$= -\frac{9}{2}u - \frac{9}{4}\sin 2u\Big]_{\pi/2}^{0}$$

$$= -\left(-\frac{9}{2}\left(\frac{\pi}{2}\right)-\frac{9}{4}\sin\pi\right)$$

$$= \frac{9\pi}{4}$$

3. **[9 marks]**

(a) If $5A = 1 - A^2$ find A^{-1} in terms of $mA + nI$ where m and n are scalars. **[3]**

$$5A = 1 - A^2$$
$$5A^{-1}A = A^{-1}(1 - A^2)$$
$$5I = A^{-1} - A^{-1}A^2$$
$$5I = A^{-1} - A$$
$$5I + A = A^{-1}$$
$$\therefore\ m = 1 \quad n = 5$$

(b) Matrices A and B are both 2x2 matrices and $A = BA - B + B^2$

Determine A if $B = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$. **[3]**

$$A - BA = -B + B^2$$
$$(I - B)A = -B + B^2$$
$$A = (I - B)^{-1}(-B + B^2)$$
$$A = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}\right)^{-1}\left(\begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}\right)$$
$$= \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}^{-1}\left(\begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 2 & -2 \end{bmatrix}\right)$$
$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} -2 & 2 \\ -4 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix}$$

(c) Determine a relationship between a and b given $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ is its own inverse. **[3]**

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$a^2 + b^2 = 1$$
$$ab + ab = 0 \rightarrow 2ab = 0$$
if $a = 0 \quad b = \pm 1$

if $b = 0 \quad a = \pm 1$

4. **[5 marks]**

Use proof by induction to show that $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$.

1 For $n=1$ $\quad$ LHS $= \frac{1}{1\times 2} = \frac{1}{2}$ $\quad$ RHS $= \frac{1}{1+1} = \frac{1}{2}$ ✓

LHS = RHS $\quad \therefore$ true for $n=1$

2 Assume true for $n=k$, so must show true for $n=k+1$ ✓

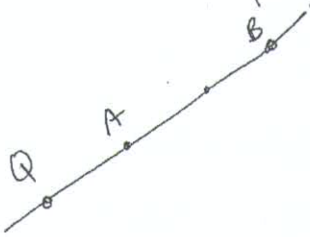
ie $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)} = \frac{k+1}{(k+1)+1}$

$$\begin{aligned}
\text{LHS} &= \frac{1}{1\times 2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1}+\frac{1}{(k+1)(k+2)} \quad ✓ \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \quad ✓ \\
&= \frac{(k+1)(k+1)}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2} \\
&= \frac{k+1}{(k+1)+1} \quad ✓ \\
&= \text{RHS}
\end{aligned}$$

So by mathematical induction proposition holds.

5. **[4 marks]**

The point Q divides the line BA externally in the ratio 3:1 . Given that Q has position vector $\begin{pmatrix} 2 \\ -\frac{1}{2} \\ -1 \end{pmatrix}$ and A $\begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$, determine the position vector of B.



$$\overrightarrow{OQ} = \overrightarrow{OA} + \tfrac{1}{2}\overrightarrow{BA}$$

$$\overrightarrow{OQ} = \overrightarrow{OA} + \tfrac{1}{2}(\overrightarrow{BO} + \overrightarrow{OA})$$

$$\overrightarrow{OQ} = \tfrac{3}{2}\overrightarrow{OA} + \tfrac{1}{2}\overrightarrow{BO}$$

$$\overrightarrow{OQ} - \tfrac{3}{2}\overrightarrow{OA} = \tfrac{1}{2}\overrightarrow{BO}$$

$$\overrightarrow{OB} = 2\left(\tfrac{3}{2}\overrightarrow{OA} - \overrightarrow{OQ}\right)$$

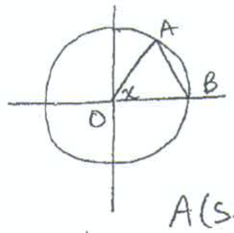
$$\overrightarrow{OB} = 2\left[\begin{pmatrix} 3/2 \\ -9/2 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ -1/2 \\ -1 \end{pmatrix}\right]$$

$$\overrightarrow{OB} = 2\begin{pmatrix} -1/2 \\ -4 \\ -2 \end{pmatrix}$$

$$\overrightarrow{OB} = \begin{pmatrix} -1 \\ -8 \\ -4 \end{pmatrix}$$

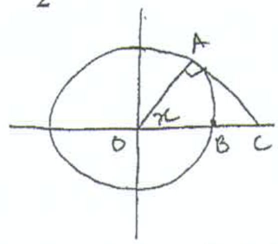
6. **[6 marks]**

(a) Establish the inequalities: $x\cos x < \sin x < x$ for $0 < x < \dfrac{\pi}{2}$ using ideas related to the unit circle. **[4]**



unit circle

$r = 1$



$A(\text{sector } OAB) > A(\triangle OAB)$

$\frac{1}{2}r^2 x > \frac{1}{2}r^2 \sin x$

$x > \sin x$

$A(\text{sector } OAB) < A(\triangle OAC)$

$\frac{1}{2}r^2 x < \frac{1}{2}r \tan x \, r$

$x < \tan x$

$x < \dfrac{\sin x}{\cos x}$

$x\cos x < \sin x$

Hence $x\cos x < \sin x < x$

(b) Use the above result to establish $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$. **[2]**

$x\cos x < \sin x < x$

$\cos x < \dfrac{\sin x}{x} < 1$

$\lim\limits_{x\to 0}$ $1 < \dfrac{\sin x}{x} < 1$

$\therefore \lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$

END OF SECTION ONE

School
Mock Examination, 2011

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Question/Answer Booklet

MATHEMATICS
3C/3D Specialist

Section Two:
Calculator-assumed

Name: Solutions

Time allowed for this section
Reading time before commencing work: 10 minutes
Working time for this section: 100 minutes

Material required/recommended for this section
To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators satisfying the conditions set by the Curriculum Council for this examination

Important note to candidates
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Structure of this paper

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Section Two Calculator-assumed	10	10	100	80
				120

Question	1	2	3	4	5	6	7	8	9	10	Total
Marks	6	8	8	8	9	8	6	8	12	7	80
Awarded											

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3. It is recommended that you **do not use pencil** except in diagrams.

Section Two: Calculator-assumed **(80 Marks)**

This section has **ten (10)** questions. Answer **all** questions. Write your answers in the space provided.

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Suggested working time for this section is 100 minutes.

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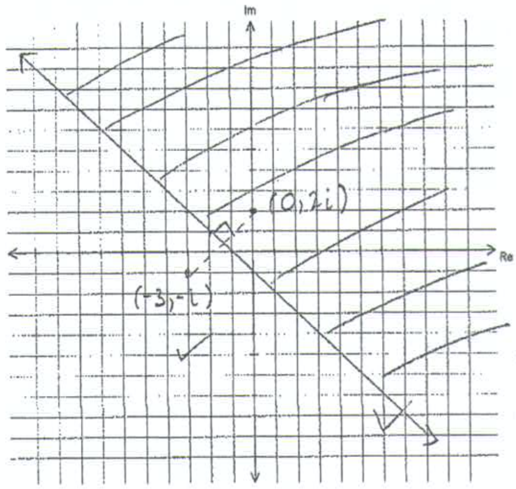
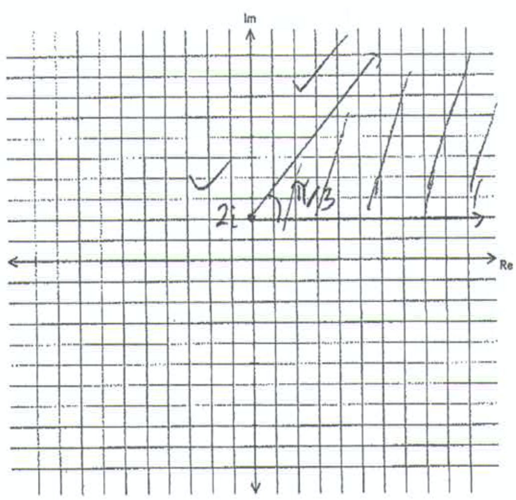
1. **[6 marks]**

On the Argand diagrams below, plot the locus of z defined by:

(a) $|z-2i| \le |z+3+i|$ [3]



(b) $\frac{\pi}{3} \le \arg(z-2i)$ [3]



2. **[8 marks]**

Arrivals at Perth Airport have the choice of 3 taxi companies; Swift, Axis or Duck. On first arrival each company has an equal chance of being selected by a customer. If they use Swift on the first arrival there is a 0.8 probability they will use it next time, 0.1 probability they will switch to Axis and 0.1 that they will switch to Duck. If they use Axis on the first arrival there is a 0.6 probability they will use it next time, 0.3 probability they will switch to Swift and a 0.1 probability they will switch to Duck. If they use Duck on the first arrival then there is a 0.5 probability they will use it next time, a 0.2 probability they will switch to Swift and 0.3 probability they will switch to Axis.

(a) Express this as a transition matrix **[2]**

$$\begin{array}{cc} \text{to} & \begin{array}{ccc} S & A & D \end{array} \quad \text{from} \\ \begin{array}{c} S \\ A \\ D \end{array} & T = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.3 & 0.6 & 0.1 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} \end{array}$$

✓✓

(b) If an arrival uses Duck on the first visit, what is the probability they will use Duck on the third visit? **[2]**

$$T^2 = \begin{array}{c} \\ S \\ A \\ D \end{array} \begin{array}{c} \begin{array}{ccc} S & A & D \end{array} \\ \begin{bmatrix} 0.69 & 0.17 & 0.14 \\ 0.44 & 0.42 & 0.14 \\ 0.35 & 0.35 & 0.3 \end{bmatrix} \end{array}$$ ✓

0.3 ✓

(c) What is the probability that an arrival uses Duck on all three visits? **[2]**

$$\frac{1}{3} \times 0.5 \times 0.5 = \frac{1}{12}$$

✓ ✓

(d) What is the probability that an arrival doesn't use the same company three times in three visits? **[2]**

$$1 - \frac{1}{3} \times 0.5 \times 0.5 - \frac{1}{3} \times 0.8 \times 0.8 - \frac{1}{3} \times 0.1 \times 0.1$$ ✓

$$= \frac{7}{10}$$ ✓

3. **[8 marks]**

Let Matrix $A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$

(a) Find A^2, A^3 and A^4 **[1]**

$A^2 = \begin{bmatrix} 4 & 0 \\ -3 & 1 \end{bmatrix}$ $A^3 = \begin{bmatrix} 8 & 0 \\ -7 & 1 \end{bmatrix}$ $A^4 = \begin{bmatrix} 16 & 0 \\ -15 & 1 \end{bmatrix}$ ✓

(b) State a conjecture for A^n, $n \in \mathbb{R}$ **[2]**

$A^n = \begin{bmatrix} 2^n & 0 \\ 1-2^n & 1 \end{bmatrix}$ ✓

(c) Prove your conjecture is true using mathematical induction **[5]**

1 $n=1$ $A^1 = \begin{bmatrix} 2^1 & 0 \\ 1-2^1 & 1 \end{bmatrix}$

LHS $= \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$ RHS $= \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$ ✓

$\therefore$ true for $n=1$

2 Assume true for $n=k$, must show true for $n=k+1$

ie $A^{k+1} = \begin{bmatrix} 2^{k+1} & 0 \\ 1-2^{k+1} & 1 \end{bmatrix}$ ✓

$$
\begin{aligned}
\text{LHS} &= A^{k+1} \\
&= A^k A \quad ✓ \\
&= \begin{bmatrix} 2^k & 0 \\ 1-2^k & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2^k 2 & 0 \\ 2(1-2^k)-1 & 1 \end{bmatrix} \quad ✓ \\
&= \begin{bmatrix} 2^{k+1} & 0 \\ 1-2^{k+1} & 1 \end{bmatrix} \\
&= \text{RHS} \quad ✓
\end{aligned}
$$

$\therefore$ By mathematical induction the conjecture holds.

4. **[8 marks]**

Given $x = \sin 2t$ and $y = 2\cos t$

(a) Show that the parametric equations can be used to form the Cartesian equation $4x^2 + y^4 = 4y^2$. **[4]**

$$4x^2 = 4(\sin 2t)^2$$
$$= 4(2\sin t\cos t)^2$$
$$= 4(4\sin^2 t\cos^2 t)$$
$$= 16\sin^2 t\cos^2 t$$

$$y^4 = (2\cos t)^4$$
$$= 16\cos^4 t$$

$$4y^2 = 4(2\cos t)^2$$
$$= 4(4\cos^2 t)$$
$$= 16\cos^2 t$$

$$4x^2 + y^4 = 16\sin^2 t\cos^2 t + 16\cos^4 t$$
$$= 16\cos^2 t(\sin^2 t + \cos^2 t)$$
$$= 16\cos^2 t$$
$$= 4y^2$$

(b) Find the equation of the tangent to the curve when $t = \frac{\pi}{4}$. **[4]**

$$4x^2 + y^4 = 4y^2$$

$$x = \sin 2(\tfrac{\pi}{4})$$
$$= 1$$
$$y = 2\cos(\tfrac{\pi}{4})$$
$$= \sqrt{2}$$

$$8x + 4y^3\frac{dy}{dx} = 8y\frac{dy}{dx}$$

$$8(1) + 4(\sqrt{2})^3\frac{dy}{dx} = 8\sqrt{2}\frac{dy}{dx}$$

$$8 = \frac{dy}{dx}(8\sqrt{2} - 8\sqrt{2})$$

$$8 = \frac{dy}{dx}(0)$$

$\frac{dy}{dx}$ = doesn't exist.

$\frac{dx}{dy} = 0$ $\therefore x = 1$

5. **[9 marks]**

The Harpy Eagle, native to South America, hunts usually in late afternoon. A Harpy Eagle flies with a constant velocity of $\langle -3 \quad 2 \quad -1 \rangle$ metres per minute. At 5:30 pm, the bird has position vector $\langle 240 \quad 76 \quad 180 \rangle$.

Determine:

(a) The position vector of the bird at 5:50 pm **[1]**

$\langle 240, 76, 180 \rangle + 20\langle -3, 2, -1 \rangle$

$= \langle 180, 116, 160 \rangle$

(b) When the bird is 150 m from its position at 5:30 pm **[2]**

$|\langle -3, 2, -1 \rangle| t = 150$

$\sqrt{14}\ t = 150$

$t = 40 \cdot 10$ mins

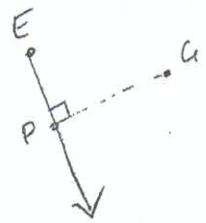
$\therefore$ 6·10 pm.

(c) When the bird crosses the X-Y plane **[1]**

$180 - t = 0$

$t = 180$ mins $\quad \therefore$ 8:30pm

(d) The closest distance between the bird and a sleeping goat on a hill with position vector $\langle 120 \quad 100 \quad 160 \rangle$ **[5]**



$\vec{EG} = \vec{EO} + \vec{OG}$

$= \langle -240, -76, -180 \rangle + \langle 120, 100, 160 \rangle$

$= \langle -120, 24, -20 \rangle$

$\vec{GE} = \langle 120, -24, 20 \rangle$

$\vec{GP} = \vec{GE} + \vec{EP}$

$= \langle 120, -24, 20 \rangle + t\langle -3, 2, -1 \rangle$

$\vec{GP} \cdot \langle -3, 2, -1 \rangle = 0$

$\langle 120 - 3t, -24 + 2t, 20 - t \rangle \cdot \langle -3, 2, -1 \rangle = 0$

$-360 + 9t - 48 + 4t - 20 + t = 0$

$14t - 428 = 0$

$t = 30.57$

$|\vec{GP}| = |\langle 120, -24, 20 \rangle + 30.5\langle -3, 2, -1 \rangle|$

$= \sqrt{2291.42..}$

$= 47.87$ m

6. **[8 marks]**

The difference between high and low tides in a Northwest port is 4 metres. The water level of the port, in x metres, is given by:

$\frac{d^2x}{dt^2} = -\frac{1}{4}x$, where t is measured in hours from the time of the high tide.

(a) Calculate the amplitude and the period of the motion of the tides. **[2]**

$A = 2m$ ✓ $T = \frac{2\pi}{n} = \frac{2\pi}{1/2} = 4\pi$ ✓

(b) Write x as a function of t. **[2]**

$x = 2\sin\frac{1}{2}t$ ✓ ✓

(c) A ship can enter or leave the port as long as there is at least 3 metres of water above the low tide mark. If the low tide takes place at 9:30am, what is the latest time a ship can leave the port on that day? **[4]**

$1 = 2\sin\frac{1}{2}t$ ✓

$\frac{1}{2} = \sin\frac{1}{2}t$

$\frac{1}{2}t = \frac{\pi}{6}$

$t = \frac{\pi}{3}$ ✓

$t = 1.047$

Low tide at 9:30am

median mark + $\frac{\pi}{2}$ $(4\pi \div 8)$

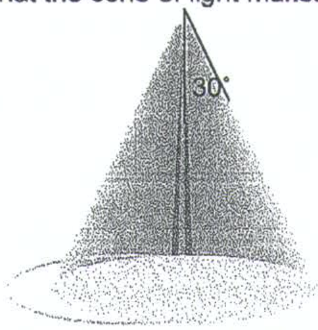
= 11·04 am ✓

11·04 am + 1hr 3mins

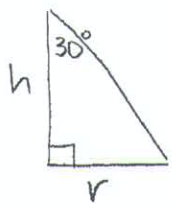
= 12·07 am ✓

7. **[6 marks]**

A helicopter is descending towards level ground. It has a bright light directed vertically downwards such that the cone of light makes a circle on the ground.



The cone has a semi vertical angle of 30°, as shown . The helicopter descends at a constant rate of 3 m/sec. At what rate is the area of the circle changing when the helicopter is 12 m above the ground?



$\frac{dh}{dt} = 3 \text{ m/s}$ ✓

$\tan 30 = \frac{r}{h}$

$\frac{1}{\sqrt{3}} = \frac{r}{h}$

$h = \sqrt{3}\, r$ ✓

$\frac{dh}{dt} = \sqrt{3}\,\frac{dr}{dt}$

$3 = \sqrt{3}\,\frac{dr}{dt}$

$\frac{dr}{dt} = \sqrt{3}$ ✓

$A = \pi r^2$ ✓

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

$= 2\pi\left(\frac{h}{\sqrt{3}}\right)(\sqrt{3})$ ✓

$= 2\pi h$

$= 2\pi(12)$

$= 24\pi \text{ m}^2/\text{s}$ ✓

8. **[8 marks]**

(a) Show that the line L whose vector equation is $\underset{\sim}{r} = -3\underset{\sim}{i} - \underset{\sim}{j} + 6\underset{\sim}{k} + \lambda(\underset{\sim}{i} - 5\underset{\sim}{j} + \underset{\sim}{k})$ is parallel to the plane Π_1 whose vector equation is $\underset{\sim}{r}.(\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}) = 12$. **[2]**

$\underset{\sim}{r}$ must be perpendicular to $\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}$ for $\underset{\sim}{r}$ to be parallel to Π_1 ✓

$$(\underset{\sim}{i} - 5\underset{\sim}{j} + \underset{\sim}{k}) \cdot (\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}) = 1 - 5 + 4$$
$$= 0 \quad ✓$$

(b) Find the equation of the plane Π_2 that contains the line L and is parallel to Π_1. **[2]**

$$\underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{a} \cdot \underset{\sim}{n}$$
$$\underset{\sim}{r} \cdot (\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}) = (-3\underset{\sim}{i} - \underset{\sim}{j} + 6\underset{\sim}{k}) \cdot (\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}) \quad ✓$$
$$= -3 - 1 + 24$$
$$\underset{\sim}{r} \cdot (\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}) = 20 \quad ✓$$

(c) Find the distance of Π_1 and Π_2 from the origin and hence, or otherwise, determine the distance between the planes. **[4]**

✓ ✓

$$\text{distance} = \frac{|1 \times 0 + 1 \times 0 + 4 \times 3 - 20|}{\sqrt{1^2 + 1^2 + 4^2}} \quad ✓$$

$$= \frac{8}{\sqrt{18}} \quad ✓$$

9. **[12 marks]**

Let $y = \cos\theta + i\sin\theta$

(a) Show that $\frac{dy}{d\theta} = iy$ [2]

$$\frac{dy}{d\theta} = -\sin\theta + i\cos\theta$$
$$= i^2\sin\theta + i\cos\theta$$
$$= i(\cos\theta + i\sin\theta)$$
$$= iy$$

(b) Hence show, using integration, that $y = e^{i\theta}$. [2]

$$\frac{1}{y}\,dy = i\,d\theta$$
$$\ln|y| = i\theta + c$$
$$y = e^{i\theta + c}$$
$$y = e^{i\theta}e^{c} \quad \text{when } \theta = 0 \quad y = 1$$
$$1 = e^{0}e^{c}$$
$$e^{c} = 1$$
$$\therefore\ y = e^{i\theta}$$

(c) Use this result to deduce de Moivre's theorem. [2]

$$y = e^{i\theta}$$
$$(y)^n = (e^{i\theta})^n$$
$$y^n = e^{in\theta}$$
$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Question 9 continued....

(d) (i) Given $\dfrac{\sin 6\theta}{\sin\theta} = a\cos^5\theta + b\cos^3\theta + c\cos\theta$, where $\sin\theta \neq 0$, use de Moivre's theorem with n = 6 to find the values of the constants a, b, and c. **[5]**

$\text{cis}\, 6\theta = (\text{cis}\,\theta)^6$

✓ $\cos 6\theta + i\sin 6\theta = (\cos\theta + i\sin\theta)^6$ expand calculator

$\sin 6\theta = 6(\cos\theta)^5\sin\theta + 6\cos\theta(\sin\theta)^5 - 20(\cos\theta)^3(\sin\theta)^3$

✓ $= 6\cos^5\theta\sin\theta + 6\cos\theta(1-\cos^2\theta)(1-\cos^2\theta)\sin\theta - 20\cos^3\theta(1-\cos^2\theta)\sin\theta$

$= 6\cos^5\theta\sin\theta + 6\cos\theta\sin\theta - 12\cos^3\theta\sin\theta + 6\cos^5\theta\sin\theta$

✓ $-20\cos^3\theta\sin\theta + 20\cos^5\theta\sin\theta$

✓ $\sin 6\theta = 32\cos^5\theta\sin\theta - 32\cos^3\theta\sin\theta + 6\cos\theta\sin\theta$

$\dfrac{\sin 6\theta}{\sin\theta} = 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta$

✓ $\therefore a = 32 \quad b = -32 \quad c = 6$

(ii) Hence deduce the value of $\lim\limits_{\theta\to 0}\dfrac{\sin 6\theta}{\sin\theta}$ **[1]**

$\lim\limits_{\theta\to 0}\dfrac{\sin 6\theta}{\sin\theta} = 32 - 32 + 6$

$= 6$ ✓

10. **[7 marks]**

Police Forensic Investigatiors are called late at night to investigate a murdered person in a suburban house. To get an idea of when the person died, the investigators use Newton's Law of Cooling which states that the rate of change of the temperature of a body is proportional to the difference between its own temperature and the ambient temperature (the temperature of the surroundings). The investigators note the body's temperature when they arrived at 3:15am was 17.4 $°C$ and at 4:15am was 15.0 $°C$. To estimate the time of death, the investigators assume the room temperature that night remained a constant 10 $°C$ and that the person's body had a temperature of 37.0 $°C$ at the time of death. Use Newton's Law of Cooling and the supplied information to estimate the time of death to the nearest 5 mins.

$$\frac{d\theta}{dt} = k(\theta - 10) \checkmark$$

$$\int \frac{1}{\theta - 10}\, d\theta = \int k\, dt \checkmark$$

$$\ln|\theta - 10| = kt + c$$

$$\theta - 10 = e^{kt+c}$$

$$\theta = 10 + e^{kt+c}$$

$$\theta = 10 + e^{c}e^{kt} \checkmark$$

let $t = 0$ be 3.15am

$$\theta = 17.4 \checkmark$$

$$17.4 = 10 + e^{c}e^{k(0)}$$

$$e^{c} = 7.4$$

$$\theta = 10 + 7.4e^{kt}$$

$$15 = 10 + 7.4e^{k(1)}$$

$$\frac{5}{7.4} = e^{k}$$

$$k = -0.392042 \checkmark$$

$$\theta = 10 + 7.4e^{-0.392t}$$

$$37 = 10 + 7.4e^{-0.392t}$$

$$\ln\left|\frac{27}{7.4}\right| = -0.392t \checkmark$$

$$t = -3.3\text{hrs} \approx 3\text{hrs } 20\text{mins}$$

$\therefore$ 11.55 pm ✓

END OF SECTION TWO